

BINOMIAL COEFFICIENT–HARMONIC SUM IDENTITIES ASSOCIATED TO SUPERCONGRUENCES

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ABSTRACT. We establish two binomial coefficient–generalized harmonic sum identities using the partial fraction decomposition method. These identities are a key ingredient in the proofs of numerous supercongruences. In particular, in other works of the author, they are used to establish modulo p^k ($k > 1$) congruences between truncated generalized hypergeometric series, and a function which extends Greene’s hypergeometric function over finite fields to the p -adic setting. A specialization of one of these congruences is used to prove an outstanding conjecture of Rodriguez-Villegas which relates a truncated generalized hypergeometric series to the p -th Fourier coefficient of a particular modular form.

1. INTRODUCTION AND STATEMENT OF RESULTS

For non-negative integers i and n , we define the generalized harmonic sum, $H_n^{(i)}$, by

$$H_n^{(i)} := \sum_{j=1}^n \frac{1}{j^i}$$

and $H_0^{(i)} := 0$. In [3] Chu proves the following binomial coefficient-generalized harmonic sum identity using the partial fraction decomposition method. If n is a positive integer, then

$$\sum_{k=1}^n \binom{n+k}{k}^2 \binom{n}{k}^2 \left[1 + 2kH_{n+k}^{(1)} + 2kH_{n-k}^{(1)} - 4kH_k^{(1)} \right] = 0. \quad (1.1)$$

This identity had previously been established using the WZ method [1] and was used by Ahlgren and Ono in proving the Apéry number supercongruence [2].

In [4], [5] the author establishes various supercongruences between truncated generalized hypergeometric series, and a function which extends Greene’s hypergeometric function over finite fields to the p -adic setting. Specifically, let p be an odd prime and let $n \in \mathbb{Z}^+$. For $1 \leq i \leq n+1$, let $\frac{m_i}{d_i} \in \mathbb{Q} \cap \mathbb{Z}_p$ such that $0 < \frac{m_i}{d_i} < 1$. Let $\Gamma_p(\cdot)$ denote Morita’s p -adic gamma function, $\lfloor x \rfloor$ denote the greatest integer less than or equal to x and $\langle x \rangle$ denote the fractional part of x , i.e. $x - \lfloor x \rfloor$. Then define

$$\begin{aligned} & {}_{n+1}G\left(\frac{m_1}{d_1}, \frac{m_2}{d_2}, \dots, \frac{m_{n+1}}{d_{n+1}}\right)_p \\ &:= \frac{-1}{p-1} \sum_{j=0}^{p-2} \left((-1)^j \Gamma_p\left(\frac{j}{p-1}\right) \right)^{n+1} \prod_{i=1}^{n+1} \frac{\Gamma_p\left(\langle \frac{m_i}{d_i} - \frac{j}{p-1} \rangle\right)}{\Gamma_p\left(\frac{m_i}{d_i}\right)} (-p)^{-\lfloor \frac{m_i}{d_i} - \frac{j}{p-1} \rfloor}. \end{aligned}$$

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Note that when $p \equiv 1 \pmod{d_i}$ this function recovers Greene's hypergeometric function over finite fields. For a complex number a and a non-negative integer n let $(a)_n$ denote the rising factorial defined by

$$(a)_0 := 1 \quad \text{and} \quad (a)_n := a(a+1)(a+2) \cdots (a+n-1) \text{ for } n > 0.$$

Then, for complex numbers a_i, b_j and z , with none of the b_j being negative integers or zero, we define the truncated generalized hypergeometric series

$${}_rF_s \left[\begin{matrix} a_1, & a_2, & a_3, & \dots, & a_r \\ & b_1, & b_2, & \dots, & b_s \end{matrix} \middle| z \right]_m := \sum_{n=0}^m \frac{(a_1)_n (a_2)_n (a_3)_n \cdots (a_r)_n}{(b_1)_n (b_2)_n \cdots (b_s)_n} \frac{z^n}{n!}.$$

An example of one the supercongruence results from [5] is the following theorem.

Theorem 1.1 ([5] Thm. 2.7). *Let $r, d \in \mathbb{Z}$ such that $2 \leq r \leq d-2$ and $\gcd(r, d) = 1$. Let p be an odd prime such that $p \equiv \pm 1 \pmod{d}$ or $p \equiv \pm r \pmod{d}$ with $r^2 \equiv \pm 1 \pmod{d}$. If $s(p) := \Gamma_p(\frac{1}{d})\Gamma_p(\frac{r}{d})\Gamma_p(\frac{d-r}{d})\Gamma_p(\frac{d-1}{d})$, then*

$${}_4G\left(\frac{1}{d}, \frac{r}{d}, 1 - \frac{r}{d}, 1 - \frac{1}{d}\right)_p \equiv {}_4F_3\left[\begin{matrix} \frac{1}{d}, & \frac{r}{d}, & 1 - \frac{r}{d}, & 1 - \frac{1}{d} \\ & 1, & 1, & 1 \end{matrix} \middle| 1\right]_{p-1} + s(p)p \pmod{p^3}.$$

A specialization of this congruence is used to prove an outstanding supercongruence conjecture of Rodriguez-Villegas, which relates a truncated generalized hypergeometric series to the p -th Fourier coefficient of a particular modular form [4],[6]. Similar results to Theorem 1.1 exist for ${}_4G$ with other parameters, and also ${}_2G$ and ${}_3G$.

The main results of the current paper, Theorems 1.2 and 1.3 below, are two binomial coefficient-generalized harmonic sum identities which factor heavily into the proofs of all the ${}_4G$ congruences. Taking particular values for n, m, l, c_1 and c_2 in these identities allows the vanishing of certain terms in the proofs. Note that letting $m = n$ in Theorem 1.2 recovers (1.1).

Theorem 1.2. *Let m, n be positive integers with $m \geq n$. Then*

$$\begin{aligned} \sum_{k=0}^n \binom{m+k}{k} \binom{m}{k} \binom{n+k}{k} \binom{n}{k} & \left[1 + k \left(H_{m+k}^{(1)} + H_{m-k}^{(1)} + H_{n+k}^{(1)} + H_{n-k}^{(1)} - 4H_k^{(1)} \right) \right] \\ & + \sum_{k=n+1}^m (-1)^{k-n} \binom{m+k}{k} \binom{m}{k} \binom{n+k}{k} / \binom{k-1}{n} = (-1)^{m+n}. \end{aligned}$$

Theorem 1.3. *Let l, m, n be positive integers with $l > m \geq n \geq \frac{l}{2}$ and $c_1, c_2 \in \mathbb{Q}$ some constants. Then*

$$\begin{aligned} & \sum_{k=0}^n \binom{m+k}{k} \binom{m}{k} \binom{n+k}{k} \binom{n}{k} \left\{ \left[1 + k \left(H_{m+k}^{(1)} + H_{m-k}^{(1)} + H_{n+k}^{(1)} + H_{n-k}^{(1)} - 4H_k^{(1)} \right) \right] \right. \\ & \quad \cdot \left[c_1 \left(H_{k+n}^{(1)} - H_{k+l-n-1}^{(1)} \right) + c_2 \left(H_{k+m}^{(1)} - H_{k+l-m-1}^{(1)} \right) \right] - k \left[c_1 \left(H_{k+n}^{(2)} - H_{k+l-n-1}^{(2)} \right) \right. \\ & \quad \left. \left. + c_2 \left(H_{k+m}^{(2)} - H_{k+l-m-1}^{(2)} \right) \right] \right\} + \sum_{k=n+1}^m (-1)^{k-n} \binom{m+k}{k} \binom{m}{k} \binom{n+k}{k} / \binom{k-1}{n} \\ & \quad \cdot \left[c_1 \left(H_{k+n}^{(1)} - H_{k+l-n-1}^{(1)} \right) + c_2 \left(H_{k+m}^{(1)} - H_{k+l-m-1}^{(1)} \right) \right] = 0. \end{aligned}$$

The remainder of this paper is spent proving Theorems 1.2 and 1.3.

2. PROOFS

We first develop two algebraic identities of which the binomial coefficient-harmonic sum identities are limiting cases.

Theorem 2.1. *Let x be an indeterminate and let m, n positive integers with $m \geq n$. Then*

$$\begin{aligned} & \sum_{k=0}^n \binom{m+k}{k} \binom{m}{k} \binom{n+k}{k} \binom{n}{k} \\ & \quad \cdot \left\{ \frac{-k}{(x+k)^2} + \frac{1 + k \left(H_{m+k}^{(1)} + H_{m-k}^{(1)} + H_{n+k}^{(1)} + H_{n-k}^{(1)} - 4H_k^{(1)} \right)}{x+k} \right\} \\ & + \sum_{k=n+1}^m \frac{(-1)^{k-n}}{x+k} \binom{m+k}{k} \binom{m}{k} \binom{n+k}{k} / \binom{k-1}{n} = \frac{x(1-x)_n(1-x)_m}{(x)_{n+1}(x)_{m+1}}. \end{aligned} \quad (2.1)$$

Proof. Using partial fraction decomposition we can write

$$f(x) := \frac{x(1-x)_n(1-x)_m}{(x)_{n+1}(x)_{m+1}} = \frac{A}{x} + \sum_{k=1}^n \left\{ \frac{B_k}{(x+k)^2} + \frac{C_k}{x+k} \right\} + \sum_{k=n+1}^m \frac{D_k}{x+k}$$

for some A, B_k, C_k and $D_k \in \mathbb{Q}$. We now isolate these coefficients by taking various limits of $f(x)$ as follows.

$$A = \lim_{x \rightarrow 0} x f(x) = \lim_{x \rightarrow 0} \frac{(1-x)_n(1-x)_m}{(1+x)_n(1+x)_m} = 1.$$

For $1 \leq k \leq n$,

$$\begin{aligned}
B_k &= \lim_{x \rightarrow -k} (x+k)^2 f(x) = \lim_{x \rightarrow -k} \frac{x(1-x)_n(1-x)_m}{(x)_k^2 (x+k+1)_{n-k} (x+k+1)_{m-k}} \\
&= \frac{-k(k+1)_n(k+1)_m}{(-k)_k^2 (1)_{n-k} (1)_{m-k}} \\
&= \frac{-k(k+1)_n(k+1)_m}{(-1)^{2k} k!^2 (n-k)! (m-k)!} \\
&= -k \binom{m+k}{k} \binom{m}{k} \binom{n+k}{k} \binom{n}{k},
\end{aligned}$$

and, using L'Hôpital's rule,

$$\begin{aligned}
C_k &= \lim_{x \rightarrow -k} \frac{(x+k)^2 f(x) - B_k}{x+k} \\
&= \lim_{x \rightarrow -k} \frac{d}{dx} \left[(x+k)^2 f(x) \right] \\
&= \lim_{x \rightarrow -k} \frac{d}{dx} \left[\frac{x(1-x)_n(1-x)_m}{(x)_k^2 (x+k+1)_{n-k} (x+k+1)_{m-k}} \right] \\
&= \lim_{x \rightarrow -k} \left\{ \left[\frac{(1-x)_n(1-x)_m}{(x)_k^2 (x+k+1)_{n-k} (x+k+1)_{m-k}} \right] \left[1 - x \left(\sum_{s=1}^n (-x+s)^{-1} \right. \right. \right. \\
&\quad \left. \left. \left. + \sum_{s=1}^m (-x+s)^{-1} + \sum_{s=1}^{n-k} (x+k+s)^{-1} + \sum_{s=1}^{m-k} (x+k+s)^{-1} + 2 \sum_{s=0}^{k-1} (x+s)^{-1} \right) \right] \right\} \\
&= \left[\frac{(1+k)_n(1+k)_m}{(-k)_k^2 (1)_{n-k} (1)_{m-k}} \right] \left[1 + k \left(\sum_{s=1}^n (k+s)^{-1} + \sum_{s=1}^m (k+s)^{-1} + \sum_{s=1}^{n-k} (s)^{-1} \right. \right. \\
&\quad \left. \left. + \sum_{s=1}^{m-k} (s)^{-1} + 2 \sum_{s=0}^{k-1} (-k+s)^{-1} \right) \right] \\
&= \binom{m+k}{k} \binom{m}{k} \binom{n+k}{k} \binom{n}{k} \left[1 + k \left(H_{m+k}^{(1)} + H_{m-k}^{(1)} + H_{n+k}^{(1)} + H_{n-k}^{(1)} - 4H_k^{(1)} \right) \right].
\end{aligned}$$

Similarly, for $n + 1 \leq k \leq m$,

$$\begin{aligned}
D_k &= \lim_{x \rightarrow -k} (x + k)f(x) = \lim_{x \rightarrow -k} \frac{x(1-x)_n(1-x)_m}{(x)_{n+1}(x)_k(x+k+1)_{m-k}} \\
&= \frac{-k(k+1)_n(k+1)_m}{(-k)_{n+1}(-k)_k(1)_{m-k}} \\
&= (-1)^{k-n} \binom{m+k}{k} \binom{m}{k} \binom{n+k}{k} / \binom{k-1}{n}.
\end{aligned}$$

□

Theorem 2.2. *Let x be an indeterminate and let l, m, n be positive integers with $l > m \geq n \geq \frac{l}{2}$ and $c_1, c_2 \in \mathbb{Q}$ some constants. Then*

$$\begin{aligned}
&\sum_{k=0}^n \frac{1}{x+k} \binom{m+k}{k} \binom{m}{k} \binom{n+k}{k} \binom{n}{k} \left\{ \left[c_1 \left(H_{k+n}^{(1)} - H_{k+l-n-1}^{(1)} \right) \right. \right. \\
&\quad \left. \left. + c_2 \left(H_{k+m}^{(1)} - H_{k+l-m-1}^{(1)} \right) \right] \cdot \left[\frac{x}{x+k} + k \left(H_{m+k}^{(1)} + H_{m-k}^{(1)} + H_{n+k}^{(1)} + H_{n-k}^{(1)} - 4H_k^{(1)} \right) \right] \right. \\
&\quad \left. - k \left[c_1 \left(H_{k+n}^{(2)} - H_{k+l-n-1}^{(2)} \right) + c_2 \left(H_{k+m}^{(2)} - H_{k+l-m-1}^{(2)} \right) \right] \right\} \\
&+ \sum_{k=n+1}^m \frac{(-1)^{k-n}}{x+k} \binom{m+k}{k} \binom{m}{k} \binom{n+k}{k} / \binom{k-1}{n} \\
&\quad \cdot \left[c_1 \left(H_{k+n}^{(1)} - H_{k+l-n-1}^{(1)} \right) + c_2 \left(H_{k+m}^{(1)} - H_{k+l-m-1}^{(1)} \right) \right] \\
&= \frac{x(1-x)_n(1-x)_m}{(x)_{n+1}(x)_{m+1}} \left[c_1 \sum_{s=l-n}^n (-x+s)^{-1} + c_2 \sum_{s=l-m}^m (-x+s)^{-1} \right]. \quad (2.2)
\end{aligned}$$

Proof. Using partial fraction decomposition we can write

$$\begin{aligned}
f(x) &:= \frac{x(1-x)_n(1-x)_m}{(x)_{n+1}(x)_{m+1}} \left[c_1 \sum_{s=l-n}^n (-x+s)^{-1} + c_2 \sum_{s=l-m}^m (-x+s)^{-1} \right] \\
&= \frac{A}{x} + \sum_{k=1}^n \left\{ \frac{B_k}{(x+k)^2} + \frac{C_k}{x+k} \right\} + \sum_{k=n+1}^m \frac{D_k}{x+k}
\end{aligned}$$

for some A, B_k, C_k and $D_k \in \mathbb{Q}$. As in the proof of Theorem 2.1, we isolate the coefficients A, B_k, C_k and D_k by taking various limits of $f(x)$. For brevity, we first let

$$T_a^{(r)} := c_1 \sum_{s=l-n}^n (a+s)^{-r} + c_2 \sum_{s=l-m}^m (a+s)^{-r}$$

and

$$U^{(r)} := c_1 \left(H_{k+n}^{(r)} - H_{k+l-n-1}^{(r)} \right) + c_2 \left(H_{k+m}^{(r)} - H_{k+l-m-1}^{(r)} \right).$$

Then we have

$$\begin{aligned} A = \lim_{x \rightarrow 0} x f(x) &= c_1 \lim_{x \rightarrow 0} \sum_{s=l-n}^n \frac{(1-x)_n (1-x)_m}{(1+x)_n (1+x)_m (s-x)} + c_2 \lim_{x \rightarrow 0} \sum_{s=l-m}^m \frac{(1-x)_n (1-x)_m}{(1+x)_n (1+x)_m (s-x)} \\ &= c_1 \sum_{s=l-n}^n s^{-1} + c_2 \sum_{s=l-m}^m s^{-1} \\ &= c_1 \left(H_n^{(1)} - H_{l-n-1}^{(1)} \right) + c_2 \left(H_m^{(1)} - H_{l-m-1}^{(1)} \right). \end{aligned}$$

For $1 \leq k \leq n$,

$$\begin{aligned} B_k &= \lim_{x \rightarrow -k} (x+k)^2 f(x) = \lim_{x \rightarrow -k} \frac{x(1-x)_n (1-x)_m}{(x)_k^2 (x+k+1)_{n-k} (x+k+1)_{m-k}} T_{-x}^{(1)} \\ &= \frac{-k(k+1)_n (k+1)_m}{(-k)_k^2 (1)_{n-k} (1)_{m-k}} T_k^{(1)} \\ &= -k \binom{m+k}{k} \binom{m}{k} \binom{n+k}{k} \binom{n}{k} U^{(1)} \end{aligned}$$

and

$$\begin{aligned} C_k &= \lim_{x \rightarrow -k} \frac{d}{dx} \left[(x+k)^2 f(x) \right] \\ &= \lim_{x \rightarrow -k} \frac{d}{dx} \left[\frac{x(1-x)_n (1-x)_m}{(x)_k^2 (x+k+1)_{n-k} (x+k+1)_{m-k}} T_{-x}^{(1)} \right] \\ &= \lim_{x \rightarrow -k} \left\{ \left[\frac{(1-x)_n (1-x)_m}{(x)_k^2 (x+k+1)_{n-k} (x+k+1)_{m-k}} \right] \left[x T_{-x}^{(2)} + T_{-x}^{(1)} - x T_{-x}^{(1)} \right. \right. \\ &\quad \cdot \left(\sum_{s=1}^n (-x+s)^{-1} + \sum_{s=1}^m (-x+s)^{-1} + \sum_{s=1}^{n-k} (x+k+s)^{-1} \right. \\ &\quad \left. \left. \left. + \sum_{s=1}^{m-k} (x+k+s)^{-1} + 2 \sum_{s=0}^{k-1} (x+s)^{-1} \right) \right] \right\} \end{aligned}$$

$$\begin{aligned}
 &= \left[\frac{(1+k)_n(1+k)_m}{(-k)_k^2(1)_{n-k}(1)_{m-k}} \right] \left[-kT_k^{(2)} + T_k^{(1)} \left(1 + k \left(\sum_{s=1}^n (k+s)^{-1} + \sum_{s=1}^m (k+s)^{-1} \right. \right. \right. \\
 &\quad \left. \left. \left. + \sum_{s=1}^{n-k} (s)^{-1} + \sum_{s=1}^{m-k} (s)^{-1} + 2 \sum_{s=0}^{k-1} (-k+s)^{-1} \right) \right) \right] \\
 &= \binom{m+k}{k} \binom{m}{k} \binom{n+k}{k} \binom{n}{k} \\
 &\quad \cdot \left[-kU^{(2)} + \left(1 + k \left(H_{m+k}^{(1)} + H_{m-k}^{(1)} + H_{n+k}^{(1)} + H_{n-k}^{(1)} - 4H_k^{(1)} \right) \right) U^{(1)} \right].
 \end{aligned}$$

For $n+1 \leq k \leq m$,

$$\begin{aligned}
 D_k &= \lim_{x \rightarrow -k} (x+k)f(x) = \lim_{x \rightarrow -k} \frac{x(1-x)_n(1-x)_m}{(x)_{n+1}(x)_k(x+k+1)_{m-k}} T_{-x}^{(1)} \\
 &= \frac{-k(k+1)_n(k+1)_m}{(-k)_{n+1}(-k)_k(1)_{m-k}} T_k^{(1)} \\
 &= (-1)^{k-n} U^{(1)} \binom{m+k}{k} \binom{m}{k} \binom{n+k}{k} / \binom{k-1}{n}.
 \end{aligned}$$

□

Proofs of Theorems 1.2 and 1.3. Multiply both sides of (2.1) and (2.2) respectively by x and take the limit as $x \rightarrow \infty$. □

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